MATH3705 A — Test 2

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

[**2**] 1. Find the general solution of $x^2y'' - 4xy' + 6y = 0$ for $x \neq 0$.

(a) $c_1|x|^{-2} + c_2|x|^3$ (b) $c_1|x|^{-2} + c_2|x|^{-3}$ (c) $c_1|x|^{-1} + c_2|x|^{-5}$
(d) $c_1|x|^2 + c_2|x|^{-3}$ (e) $c_1|x|^2 + c_2|x|^3$

Solution: (e)

The indicial equation is

$$r^2 - 5r + 6 = 0, \Rightarrow r = 2, 3.$$

The general solution is

$$y(x) = c_1|x|^2 + c_2|x|^3.$$

[**2**] 2. Find the general solution of $x^2y'' + 5xy' + 4y = 0$ for $x \neq 0$.

(a) $c_1|x|^2 + c_2|x|^{-2}\ln|x|$ (b) $c_1|x|^{-2} + c_2|x|^2\ln|x|$ (c) $c_1|x|^2 + c_2|x|^2\ln|x|$
(d) $c_1|x|^2 + c_2|x|^{-2}$ (e) $c_1|x|^{-2} + c_2|x|^{-2}\ln|x|$

Solution: (e)

The indicial equation is

$$r^2 + 4r + 4 = 0, \Rightarrow r = -2.$$

The general solution is

$$y(x) = c_1|x|^{-2} + c_2|x|^{-2}\ln|x|.$$

[2] 3. Which of the following is a solution of $x^2y''(x) + xy'(x) + (3x^2 - 4)y(x) = 0$, for $x > 0$:

(a) $J_2(\sqrt{3}x)$ (b) $J_4(\sqrt{3}x)$ (c) $J_9(4x)$ (d) $J_{\sqrt{3}}(2x)$ (e) $J_{\sqrt{3}}(4x)$

Solution: (a). Note that $\lambda^2 = 3$ and $\nu^2 = 4$, $\Rightarrow \lambda = \sqrt{3}$ and $\nu = 2$. Hence

$$y_1(x) = J_2(\sqrt{3}x), \quad y_2(x) = Y_2(\sqrt{3}x).$$

[2] 4. Which of the following is a solution of $x^2y''(x) + xy'(x) + (x^2 - 2.25)y(x) = 0$, for $x > 0$:

(a) $J_{-1.5}(x)$ (b) $J_1(1.5x)$ (c) $Y_{2.25}(x)$ (d) $Y_1(2.25x)$ (e) $Y_1(1.5x)$

Solution: (a). Note that $\lambda^2 = 1$ and $\nu^2 = 2.25$, $\Rightarrow \lambda = 1$ and $\nu = 1.5$. Hence

$$y_1(x) = J_{1.5}(x), \quad y_2(x) = J_{-1.5}(x).$$

[5] 5. Find the first four terms of the series solution about $x = 0$ of the initial value problem

$$(x^2 + 1)y'' + y' + y = 0, \quad y(0) = 1, y'(0) = -2.$$

Solution: Since $x = 0$ is an ordinary point of this equation, we may let

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots .$$

Then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots ,$$
$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots .$$

Substitute them into the differential equation,

$$(x^2 + 1)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots) + (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)$$
$$+(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = 0,$$

i.e.,

$$(a_0 + a_1 + 2a_2) + (a_1 + 2a_2 + 6a_3)x + \cdots = 0.$$

(2 points)

We have

$$a_0 = y(0) = 1,$$
$$a_1 = y'(0) = -2,$$
$$a_0 + a_1 + 2a_2 = 0, a_2 = \frac{1}{2},$$
$$a_1 + 2a_2 + 6a_3 = 0, a_3 = \frac{1}{6},$$

The first four terms in the solution to this problem is

$$y = 1 - 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots .$$

(3 points)

[7] 6. Consider the following equation

$$xy'' + (x - 0.5)y' - 0.5y = 0$$

for $x > 0$ near $x_0 = 0$. We see that $x_0 = 0$ is a regular singular point. Let

$$y = \sum_{n=0}^{\infty} c_n(r)x^{n+r}, \quad c_0(r) = 1.$$

The recursive relation is

$$c_{n+1}(r) = \frac{-1}{n+r+1}c_n(r), n \geq 0.$$

(1) (2 points) Write down the indicial equation and solve it to determine r_1 and r_2, $r_1 \geq r_2$.

Solution: We rewrite the DE as

$$y'' + \frac{x-0.5}{x}y' - \frac{0.5}{x}y = 0.$$

Then

$$xp(x) = -0.5 + x, \quad x^2q(x) = -0.5x.$$

Thus $p_0 = -0.5, q_0 = 0$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 - 1.5r = 0. \Rightarrow r_1 = 1.5, r_2 = 0.$$

Note that $r_1 - r_2 = 1.5$, so we have Case (i).

(2) (4 points = 2+2) Solve $c_n(r_1)$ and $c_n(r_2)$.

Solution: Take $r = r_1 = 1.5$, by the recursive equation,

$$c_{n+1}(1.5) = \frac{-1}{n+2.5}c_n(r), n \geq 0.$$

$$c_1(1.5) = \frac{-1}{2.5}c_0(1.5) = \frac{-1}{2.5},$$

$$c_2(1.5) = \frac{-1}{3.5}c_1(1.5) = \frac{(-1)^2}{2.5(3.5)}, ...,$$

$$c_n(1.5) = \frac{(-1)^n}{2.5(3.5)...(n+1.5)}, \quad or \quad \frac{(-2)^n}{5(7)...(2n+3)}, \quad n \geq 1.$$

Take $r = r_2 = 0$, by the recursive equation,

$$c_{n+1}(0) = \frac{-1}{n+1}c_n(r), n \geq 0.$$

$$c_1(0) = \frac{-1}{1}c_0(0) = \frac{-1}{1},$$

$$c_2(0) = \frac{-1}{2}c_1(0) = \frac{(-1)^2}{1(2)}, ...,$$

$$c_n(0) = \frac{(-1)^n}{1(2)...(n)} = \frac{(-1)^n}{n!}, n \geq 1.$$

(3) (1 point) Find two linearly independent solutions.

Solution: By (2) we have

$$y_1 = x^{1.5} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2.5(3.5)...(n+1.5)} x^{n+1.5} \;\; or \;\; x^{1.5} + \sum_{n=1}^{\infty} \frac{(-2)^n}{5(7)...(2n+3)} x^{n+1.5}.$$

$$y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n.$$